Unsymmetrical N-Dimensional Binary Colorings

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In 1971, Engel posed a problem on coloring an *n*-dimensional binary solid. Take an *n*-dimensional hypercube with each edge of length 2. (e.g. a 2x2 square in 2-dimensions or a 2x2x2 cube in 3-dimensions.) This binary hypercube consists of 2^n unit hypercubes. Consider all possible colorings of these unit hypercubes with half of them colored white and half black (half-and-half colorings). Engel noticed that in all colorings in one, two, and three dimensions, the white half is symmetrically identical to the black half. That is, by rotations and reflections, the pattern formed by the white half can be made to correspond to the pattern formed by the black half. (See Figure 1 for two examples in three dimensions.) Engel posed the question: "Are all the binary colorings of an *n*-dimensional binary solid symmetrical?"

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Insert Figure 1 about here

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The answer is no; in higher dimensions there exist colorings in which the black and white halfs do not match. Consider the 4-dimensional coloring shown in Figure 2, where neighboring unit hypercubes are connected by lines. To see that this coloring is not symmetrical, examine the upper-left white cube. Note that all four neighbors of this cube are black. Thus, there are two separate white components. Also note that all the black cubes form a single connected component. Since the arrangements of black cubes and white cubes are topologically different, the arrangements will not correspond under any

rotations or reflections. Thus, the coloring is not symmetrical.

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Insert Figure 2 about here

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We can use a similar technique to create an unsymmetrical coloring in any dimensions above 3 by generating a coloring with disconnected white components and a connected black component. First apply an n-dimensional coordinate system to the hypercube, with each coordinate being 0 or 1. Two unit cubes will be neighbors if their positions differ in a single coordinate. Now, color the cube with coordinate (0,0,0,0,...) white. Color the cubes (1,0,0,0,...), (0,1,0,0,...), (0,0,1,0,...), ..., black; that is, color the n neighbors of the white cube black. Next, color the $n-1$ cubes with the first coordinate 1 and one other coordinate 1 black: (1,1,0,0,...), (1,0,1,0,...), (1,0,0,1,...), ... When this coloring is completed, note that black cube (1,0,0,0,...) is a neighbor to each cube in the second black set. Also note that each cube in the first set has a neighbor in the second set. Thus, the $2n-1$ black cubes form a single connected component. Finally, color $2^{n-1}-(2n-1)$ additional neighboring cubes black, resulting in half the cubes being black and forming a single connected component. At the end of this construction, we have a half-and-half coloring with a single black component and multiple white components. Thus we have created an unsymmetric coloring. Note that n must be at least 4 for $2^{n-1}-(2n-1)$ to be positive. Thus the construction fails in 3 dimensions or fewer.

Using a simple computer program I tested all 4-dimensional colorings and found 74 unique half-and-half colorings (excluding rotations and reflections.) Of these, 42 had symmetrical white and black halfs. The remaining 32 had unsymmetrical black and white halfs, and thus are counterexamples to Engel's question. Figure 3 lists 16 of the 32 unsymmetric colorings; the other 16 are formed by reversing the black/white coloring. The unit hypercubes in Figure 3 are arranged as in Figure 2.

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Insert Figure 3 about here

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Acknowledgements

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References

1. D. A. Engel, An N-Dimensional Binary Coloring Problem, *Journal of Recreational Mathematics, 4*:3, pp. 199-200, 1971.

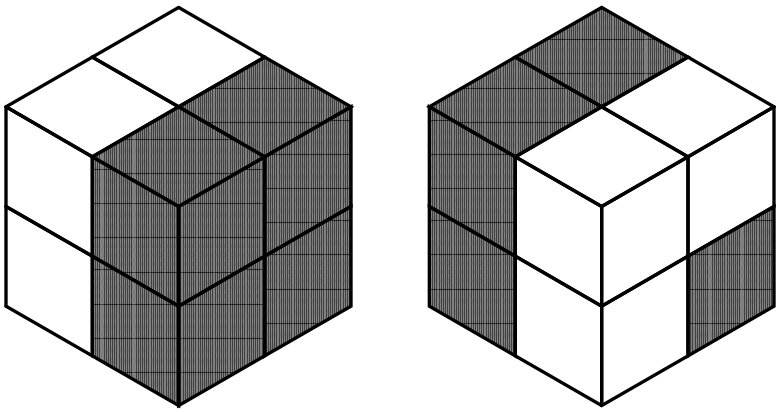

Figure 1. Two half-and-half colorings of the 3-dimensional binary cube.

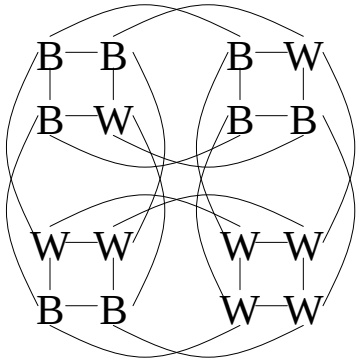

Figure 2. A unsymmetric half-and-half coloring of the 4-dimensional binary hypercube. The lines connect unit hypercubes which are neighbors in the 4-dimensional space.

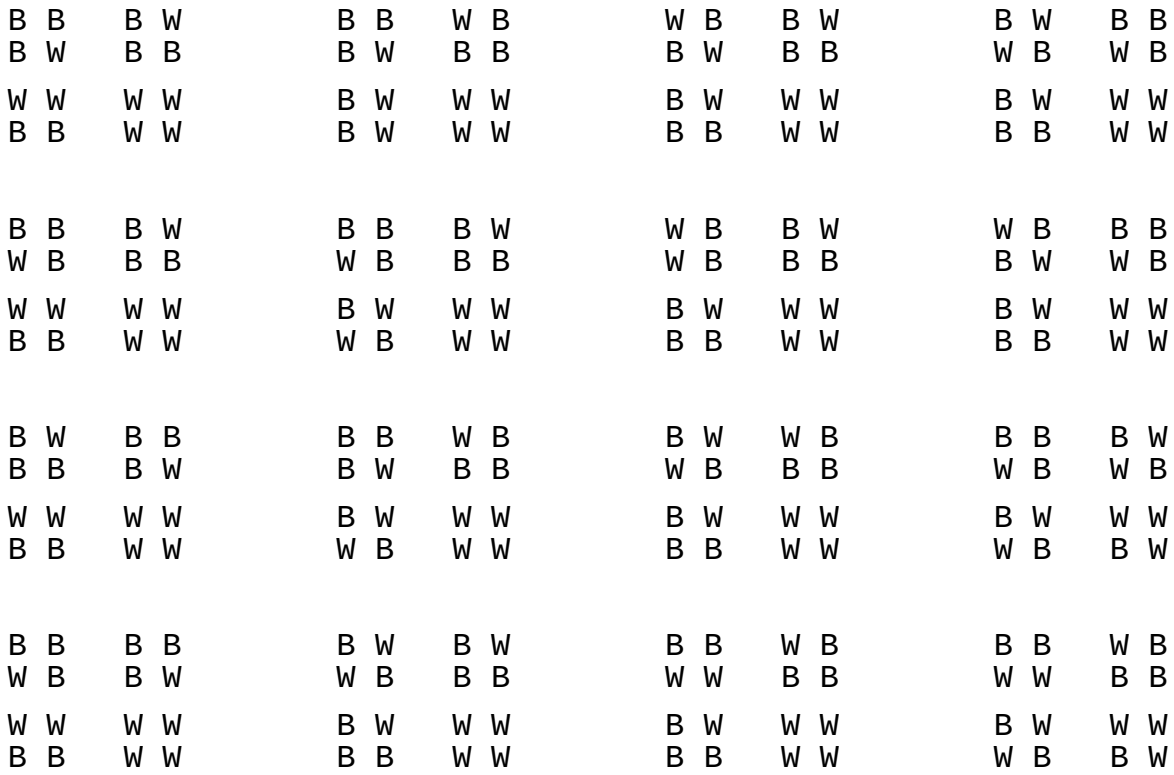

Figure 3. Sixteen of the 32 unsymmetrical half-and-half colorings of the 4-dimensional binary hypercube. (The other sixteen are formed by reversing black and white.)