An Investigation of Fractals Generated by $z \rightarrow 1/z\hat{}n + c$

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ABSTRACT

This paper explores methods of generating fractals from the mapping $z \rightarrow z^{-n} + c$ and discusses the structure of the resulting images. Lyapunov exponents and cycle periodicity show details of these fractals that are obscured by prior escape-time techniques. The resulting fractals are shown to have a hypocycloid shape. The associated Julia sets can be classified into a variety of types.

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Introduction

Several programs and papers have used escape-time methods to produce images of fractals based on the complex mapping (1), where exponent n is a positive integer.

$$z \rightarrow z^{-n} + c \tag{1}$$

However, these methods do not clearly show the fundamental structure of the fractals. This paper explores alternative methods of generating these fractals that reveal new structures. These methods are Lyapunov exponent computation and cycle periodicity checking.

Generating the fractals

Fractals have been generated from (1) using escape-time techniques, for example by Gujar et al [1] and Glynn [2]. Figure 1 shows a c-plane escape-time fractal generated by starting with an initial z value, iterating the mapping (1) with exponent n=2 and c depending on the pixel, and stopping if the magnitude of z exceeds a fixed bound. Note that the initial value z_0 should be infinity, since infinity is the critical point of (1). However instead of starting with z_0 = infinity, it is simpler to start with $z_1 = c$, which yields the same result. (A critical point of $z \rightarrow F(z)+c$ is a point where $F'(z)=0$. The role of critical points is explained in [3].)

The escape-time technique is not as satisfactory for (1) as it is for fractals such as the Mandelbrot set due to the difference in dynamics between the two mappings. For the

Mandelbrot mapping, $z \rightarrow z^2+c$, infinity is an attracting value; sufficiently large z values will diverge to infinity. However, the mapping (1) doesn't diverge to infinity since large z values get mapped back near c. As a result, looping until z is large is not a rigorous stopping condition for mapping (1); the structure of the fractal strongly depends on the stopping radius, as Glynn found in [2]. This is illustrated by Figure 2, which shows a fractal generated by the same method as Figure 1, but with a stopping radius of 10 instead of 3. This suggests that much of the structure of the fractals in Figures 1 and 2 is an artifact of the stopping criterion. The remainder of this paper examines methods of generating fractals from (1) that avoid these artifacts.

One method of displaying the fractals is to compute the Lyapunov exponents of the mapping (1). Figure 3 shows the results of this process for n=2; black points are those with a positive Lyapunov exponent, corresponding to chaotic behavior. According to [4], the Lyapunov exponent λ for the mapping $z_{n+1}=F(z_n)+c$ can be computed from:

$$\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} \ln |F'(z_n)| \tag{2}$$

A positive Lyapunov exponent indicates chaotic behavior, and a negative Lyapunov exponent indicates stable behavior. For Figure 3, the Lyapunov exponent was approximated at each point by performing 250 iterations of the mapping in order for the values to settle, and then summing (2) over the next 250 iterations. (See [4] for more details on computing Lyapunov exponents.)

The Lyapunov method reveals several new features of the fractal, as can be seen by comparing Figure 3 with Figure 2. The main difference is that the large dark circular regions of Figure 2 have disappeared; they are artifacts of the escape-time techniques. These regions correspond to areas that are mapped near infinity by (1), causing them to exceed the escape-time bound. Figure 3 is much less dense than Figure 2; it has a lace-like structure and numerous voids. Figure 3 probably shows the fundamental structure of the fractal; its shape does not depend on arbitrary values such as the escape bound.

Figure 4 shows a detail of a very small part of Figure 3; this detail reveals several interesting things. First, there are numerous stable triangular structures resembling the quartic Mandelbrot set generated by $z \to z^4 + c$. Presumably these arise because iterating z^{-2} twice results in z^4. Second, Figure 4 is roughly self-similar; zooming in on Figure 4 to any level of detail results in an image resembling Figure 4. Finally, all the unstable parts of Figure 3 appear to have small details resembling Figure 4. Thus, there doesn't seem to be any uniformly chaotic part of the fractal; there are tiny stable regions everywhere. This probably accounts for the lace-like structure of Figure 3.

A second technique for generating a fractal from (1) is to color points according to the periodicity of the resulting sequences. For each c value, we can iterate the mapping and test if the resulting sequence of z values approaches a cycle. Points leading to cycles can be colored according to the length of the cycle, and points that never enter a cycle but wander chaotically can be left black. (An algorithm for finding cycles is given in [5].) Figure 5 shows the results of this process. There is an outer gray area in which z converges to a fixed point, a central red region that converges to a 2-cycle, other regions that converge to longer cycles, and black points that apparently don't converge to cycles.

If we color periodic points white and nonperiodic points black, we obtain an image nearly identical to Figure 3. This is not surprising since stable cycles have negative Lyapunov exponents. (Mandelbrot [6] and Peitgen et al. [7] applied a similar coloring technique to $z \to \lambda(z - 1/z)$. Interestingly, the resulting image contains quadratic Mandelbrot set ''island molecules'' analogous to the quartic Mandelbrot sets that appear in Figure 4.)

Structure of the fractal

By examining fixed points, we can mathematically determine the basic shape of the fractal. Consider the c values that result in convergence to a fixed point of (1). A fixed point of (1) will satisfy

$$c=z-z^{-n} \tag{3}$$

For a stable cycle, the derivative of (1) must have magnitude less than 1; that is, $|-nz^{-n-1}|<1$ so

$$|z|>n^{1/(n+1)} \tag{4}$$

Thus, the fixed point region in the c-plane (the gray region in Figure 5) is given by (3), where z satisfies (4). (A similar analysis is applied to the Mandelbrot set by Branner in [3].)

Geometrical analysis of (3) and (4) shows that the boundary of the fixed point region forms a n+1 cusped hypocycloid. (A hypocycloid is the curve formed by rolling a smaller circle inside a larger circle and tracing a fixed point on the circumference of the smaller circle.) The radius of the outer fixed circle for the hypocycloid can be computed as $|z|+|z^{-n}|$, where z satisfies (4), resulting in a radius of $(n+1)n^{-n/(n+1)}$. The radius of the inner moving circle is $|z^{-n}|$, yielding $n^{-n/(n+1)}$. Figure 6 shows the fractal generated from $z \to z^{-4}+c$ using the Lyapunov exponent method; note the hypocycloid shape of the outer boundary.

For n=2, the 2-periodic region is a circle with radius $2^{-2/3}$; this is the red region in Figure 5. To see this, note that a point with period 2 must satisfy $(z^{-2}+c)^{-2}+c=z$, and thus $(c-z+c^2z^2)(1+cz^2-z^3)=0$. The second term corresponds to a fixed point; cancelling this term and dividing by zc yields $z^{-1}+zc=c^{-1}$. Now, for stability $|d/dz\,((z^{-2}+c)^{-2}+c)|<1$. This simplifies to $|(z^{-1}+zc)^{-1}|<2^{-2/3}$. Finally, combining the two equations yields $|c|<2^{-2/3}$, proving the 2-periodic region is a circle.

For n=3, the 2-periodic region is given by $c=\pm\sqrt{a+a^3}$ where $|a|<3^{-1/2}$. This follows from the stability condition $|d/dz\,((z^{-3}+c)^{-3}+c)|<1$, which yields $|(z^{-2}+zc)^{-1}|<3^{-1/2}$. Defining $a=(z^{-2}+zc)^{-1}$ and algebraically manipulating the 2-cycle condition $(z^{-3}+c)^{-3}+c=z$ yields the equation specifying the region. A similar derivation shows that for n=2, the 3-periodic regions are given by $c^3=-(1+a+a^2)$, where $|a|<1/2$; these are the three orange cardioid-like regions in Figure 5.

The fractals generated from (1) have rotation and reflection symmetries. As conjectured by Gujar et al in [1], the fractal generated with exponent n is (n+1)-way rotationally symmetric. This can be seen since if $f(z)=z^{-n}+c$, $g(z)=z^{-n}+e^{2\pi i/(n+1)}c$, and $h(z)=e^{2\pi i/(n+1)}z$, then $h \circ f = g \circ h$, so f and g are topologically conjugate by homeomorphism h. Therefore the parameter values c and $e^{2\pi i/(n+1)}c$ result in the same dynamic behavior and the fractal is (n+1)-way rotationally symmetric. (In other words, rotating c by $e^{2\pi i/(n+1)}$ simply rotates terms in the resulting z sequence by the same amount.) Similarly, if $f(z)=z^{-n}+c$, $g(z)=z^{-n}+\bar{c}$, and $h(z)=\bar{z}$, then f and g are topologically conjugate. Thus, c and $\bar{c}$ yield the same dynamic behavior, so the fractal is symmetrical by reflection in the real axis.

Julia sets

A z-plane Julia set can be generated from (1) by fixing c and varying the starting z value in the plane. Gujar et al have used escape-time techniques [8] to create z-plane fractals from (1). A different way to generate Julia sets that avoids escape-time problems and results in new images is to iterate each z value and examine its cyclical behavior. For example, pick a c value such that if iteration is started with $z_0=c$, the result is convergence to an attractive cycle of length 2. Call the two values in the cycle *a* and *b*. If we now iterate the mapping consisting of (1) composed with itself, which we will denote as $f^2(z)$, the cycle of length 2 becomes a fixed point, either *a* or *b*. Finally, for any z value in the plane, we can iterate $f^2(z)$, see if it converges to *a* or to *b*, and color the associated pixel one of two colors.

Figure 7 shows the result of the fixed-point coloring process applied to c=0.106+0.619i. (This c value is near the edge of the central red circle in Figure 3.) The Julia set is the boundary between the black and white regions in Figure 7, and consists of z values that don’t converge to the cycle. (Note: the exponent in (1) is n=2 for all images in this section. Julia sets with other exponents can be generated similarly.)

The Julia sets corresponding to longer cycles can be computed in a similar way. Figure 8 shows the filled-in Julia set generated from c=0.48+0.9i. The c value is in the large upper-right orange circle of Figure 5 corresponding to a cycle of length 3. Thus, most z values converge to a stable 3-cycle, and there are three fixed points of $f^3(z)$, resulting in the three colors of regions in Figure 8. Figure 9 shows the filled-in Julia set for c = 0.72+1.05i; the corresponding cycle is of length 7, resulting in 7 regions. Note that the regions never meet along an edge, but only at single points where all 7 regions meet. (This follows from a theorem [9] that the Julia set of a rational function is the frontier of the stable set associated with any fixed point. Thus, any boundary point must be on the border of every region.)

Based on experimental image generation, it appears that the Julia sets can be roughly categorized. A c value that results in convergence to a fixed point (i.e. a c value in the gray region of Figure 5) yields a Julia set that is a Cantor set. A c value that converges to a cycle of length 2 (i.e. the red region in Figure 5) results in a Julia set that is a simple closed fractal curve between two regions, as in Figure 7. As the c values approaches the boundary of the red region, the Julia sets becomes more jagged. For c values in one of the large rounded regions centrally located in an arm, touching two gray regions in Figure 5, the Julia sets are resemble Figure 8. These Julia sets are reminiscent of fractals generated by finding basins of attractions of Newton’s method [7]. For c values in one of the buds off the gray region in Figure 5, the Julia sets resemble Figure 9, with large regions with complex boundaries that pinch together at various points. Other c values corresponding to stable values result in a Julia set that is somewhat foam-like, with many rounded regions of various sizes packed together. For c values corresponding to unstable values, we were unable to generate Julia sets by the cycle-finding method, as the iteration was not cyclical. The Julia set classifications appear to generalize to exponents other than n=2.

The Julia sets generated from (1) are n-way rotationally symmetric, as conjectured by Gujar et al [8]. Since $(e^{2\pi i/n}z_0)^{-n}+c=z_0^{-n}+c$, the rotation of z_0 drops out in the first iteration. In general, the Julia sets are not symmetrical by reflection unless c is real.

Conclusions

Interesting fractals can be generated from $z\rightarrow z^{-n}+c$ by computing Lyapunov exponents and by determining the periodicity of cycles. These fractals have a detailed structure that is obscured by the escape-time method of computation. The corresponding Julia sets yield a variety of structures.

There are several avenues for future research. The equation (1) can be analyzed for noninteger or even complex exponent by the use of the Lyapunov and cycle techniques, extending the analysis of Gujar et al. The techniques of this paper can also be used with other formulas besides (1); the techniques may be especially useful for formulas such as $z\rightarrow(\bar{z})^{-n}+c$ or $z\rightarrow\cosh(z)+c$ [10] that do not diverge to infinity.

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Figure captions:

Fig. 1: Escape-time fractal generated from $z \rightarrow z^{-2}+c$. The initial z value is c; the escape condition is $|z|>3$. Dark points diverge quickly; white points are stable.

Fig. 2: Escape-time fractal generated from $z \rightarrow z^{-2}+c$. The escape condition is $|z|>10$.

Fig. 3: Lyapunov exponent fractal generated from $z \rightarrow z^{-2}+c$. Stable points (i.e. those with negative exponents) are white; unstable points are black.

Fig. 4: A very small detail of Figure 4. The coordinates are $0.546 \leq x \leq 0.547$, $0.620 \leq y \leq 0.621$. The stable points are white.

Fig. 5: Fractal generated from $z \rightarrow z^{-2}+c$, colored according to periodicity of the mapping. Short cycles are colored according to length: 1=gray, 2=red, 3=orange, 4=yellow, 5=green, 6=blue. Points that did not converge to a cycle are colored black.

Fig. 6: Lyapunov exponent fractal generated from $z \rightarrow z^{-4}+c$.

Fig. 7: Filled-in Julia set generated from $z \rightarrow z^{-2}+c$ with $c = 0.106+0.619i$. This c value is near the edge of the central red region of Figure 5 and generates a cycle of length 2. Points are colored according to the corresponding fixed point of $f^2(z)$.

Fig. 8: Filled-in Julia set generated from $z \rightarrow z^{-2}+c$ with $c = 0.48+0.9i$. This c value is in the upper right large orange region of Figure 5 and generates a cycle of length 3. Points are colored according to the fixed point of $f^3(z)$.

Fig. 9: Filled-in Julia set generated from $z \rightarrow z^{-2}+c$ with $c = 0.72+1.05i$. This point is in a small bud off the right-hand gray region of Figure 5 and generates a cycle of length 7. Points are colored according to the fixed point of $f^7(z)$.