Fractals from Simple Polynomial Composite Functions

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ABSTRACT

This paper describes a method of generating fractals by composing two simple polynomial functions. Many common fractals, such as the Mandelbrot set, the tricorn, and the forced logistic map, as well as new fractals can be generated with this technique. In many cases, the symmetry of the resulting fractal can be easily proved.

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Introduction

A common method of generating fractals is repeated iteration of a complex function: $z_{n+1}=F(z_n)$, where the function F depends on a constant c. Pickover has described a technique [1], [2] for creating the function F by composition of two simpler functions f_1 and f_2: $F(z)=f_2(f_1(z))$. This paper discusses fractals generated by composition of functions that are simple polynomials in z:

$$f_i(z)=z^{k_i}+h_i(c) \tag{1}$$

This class of functions generates a wide variety of fractals, while being easily analyzable mathematically. (This class includes the elemental polynomial processes mentioned in [3].)

Fractals in the c-plane (or parameter plane) can be generated from (1) by using common escape-time techniques. That is, for each c value, a sequence of z values is generated recursively from the following set of equations:

$$\begin{cases} z_0=0 \\ z_{2n+1}=f_1(z_{2n}) \\ z_{2n+2}=f_2(z_{2n+1}) \end{cases} \tag{2}$$

If the resulting sequence diverges, c is outside the fractal; otherwise, c is inside the fractal. Julia set fractals can be generated in the z-plane (or dynamical plane) by fixing c and varying the starting z_0 in Equation 2.

Symmetry results

In many cases, it is straightforward to determine the symmetry of fractals generated from (2). The Appendix contains proofs of these results.

Result 1: If (1) satisfies, for all c:

a) $h_1(\overline{c})=h_1(c)$ and $h_2(\overline{c})=h_2(c)$, or

b) $h_1(\overline{c})=\overline{h_1(c)}$ and $h_2(\overline{c})=\overline{h_2(c)}$

then the resulting *c*-plane fractal is symmetrical by reflection in the real axis. (The overbar indicates the complex conjugate.)

Result 2: If there are rotations r, s, and t such that (1) satisfies, for all c:

$h_1(e^{ir}c) = e^{is}h_1(c)$, $h_2(e^{ir}c) = e^{it}h_2(c)$, $e^{isk_2} = e^{it}$, and $e^{itk_1} = e^{is}$,

then the resulting *c*-plane fractal is symmetrical by rotation around the origin by r radians.

Result 3: A *z*-plane fractal (i.e. a Julia set) generated from (1) will be symmetrical by rotation around the origin by $2\pi/k_1$ radians. That is, it will have k_1-way symmetry.

Applications

As an example, the Mandelbrot set, as well as the generalization to arbitrary powers ($z_{n+1}=z_n^k+c$), can trivially be expressed as a composite function fractals by setting $f_1(z)=f_2(z)=z^k+c$ (for k an integer ≥ 2). Results 1 and 2 prove that the generalized Mandelbrot set has $k-1$ way rotational symmetry and is symmetric by reflection in the real axis. (Gujar et al. illustrated this symmetry in [4].) This symmetry can be seen by setting $r=s=t=2\pi/(k-1)$. Result 3 shows the associated Julia sets have k-way rotational symmetry, as claimed in [5] and [6].

A second example is the Mandelbar or tricorn fractal [7] and its generalizations, generated by iterating $z_{n+1}=\overline{z}_n^k+c$. By taking the complex conjugate of even z terms, this can be converted to a composite form: $f_1(z)=z^k+c$, $f_2(z)=z^k+\overline{c}$. Results 1 and 2 then show that the generalized Mandelbar set has k+1 way rotational symmetry and is symmetric by reflection in the real axis. This symmetry can be seen by setting $r = s = 2\pi/(k+1)$ and $t = -2\pi/(k+1)$. Result 3 proves that Julia sets from the Mandelbar function are k-way rotationally symmetric.

The symmetry results can be used to find fractals with a desired symmetry, such as pentagonal symmetry. (A detailed exploration of pentagonal symmetry can be found in [8].) Figures 1 through 3 show composite fractals with pentagonal symmetry. These figures were generated by applying Equation (2) 300 times for each pixel on a 2000x2000 grid. If the magnitude of z ever exceeded 10, the iteration was considered to diverge; otherwise the pixel was colored black. The formulas for these fractals were determined by choosing a formula with several undetermined parameters and then using the symmetry results to determine valid values for the parameters. For instance, suppose we want a pentagonally symmetric fractal of the form: $f_1(z)=z^{k_1}+c$, $f_2(z)=z^{k_2}+c^2$. Applying Result 2 for pentagonal symmetry, we have $r=2\pi/5$, which yields $s=2\pi/5$ and $t=4\pi/5$. It is then straightforward to solve for suitable values of k_1 and k_2, such as $k_1=3$ and $k_2=2$. Some other interesting pentagonally symmetric composite fractals are generated by:

$f_1(z)=z^6+c$, $f_2(z)=z^6+c$ (A generalized Mandelbrot set.)

$f_1(z)=z^4+c$, $f_2(z)=z^4+\bar{c}$ (A generalized Mandelbar set.)

$f_1(z)=z^4+c^3$, $f_2(z)=z^4+c^2$

$f_1(z)=z^4+c$, $f_2(z)=z^4+c^{-1}$

$f_1(z)=z^2+c^5$, $f_2(z)=z^2+|c|^2$

Note that switching f_1 and f_2 generally results in a new fractal image, but both images will have the same symmetry under Results 1 and 2.

A variant of the periodically forced logistic map discussed by Markus and Hess [9] can be expressed as a composite fractal. The logistic map $x_{n+1}=rx_n(1-x_n)$ is forced by alternating the parameter r between two values. We can use a change of variables to express this in the form of (1): $f_1(z)=z^2+Re(c)$, $f_2(z)=z^2+Im(c)$. (Re and Im are the real and imaginary parts of the complex value c.) The two real parameters to the forced logistic map are replaced by the single complex value c, but f_1 and f_2 are now real-valued functions.

One benefit of the composite fractal expression of the logistic equation is that it suggests interesting generalizations, such as using different exponents. Figure 4 shows the Lyapunov image for $f_1(z)=z^2+Re(c)$, $f_2(z)=z^3+Im(c)$. This figure was generated using techniques similar to those in [9]. For each pixel, the routine `compute_lyapunov_pixel` computed the Lyapuonv exponent lambda. Unstable pixels (with positive lambdas) were colored black. Stable pixels with large negative lambda values were colored dark and pixels with values near zero were colored white. The parameter `max` controls the total number of iterations, while `skip` controls the number of initial iterations that are skipped while the sequence stabilizes. Increasing these parameter improves the accuracy of the lambda computation, but slows down image generation.

Conclusions

Many interesting fractals can be generated from composition of simple polynomial functions. In many cases, the symmetry of the resulting fractal can be easily proved. Avenues for future work include composing more than two formulas, using more complex composition sequences (e.g. $F(z)=f_1(f_1(f_2(z)))$, or using more complex formulas.

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Appendix: Outline proofs of the results

Results 1 and 2 can be proved by comparing the two sequences of z values obtained from two parameters c and c'. That is, Equation (2) generates the sequence $\{0, z_1, z_2, z_3, \cdots\}$ from parameter c, and the sequence $\{0, z'_1, z'_2, z'_3, \cdots\}$ from parameter c'. If $|z_n| = |z'_n|$ for all n, then either both sequences converge or both sequences diverge. Thus, c is inside the fractal if and only if c' is. (A similar argument is used in [7].)

For Result 1, set $c' = \overline{c}$. In case (a), the sequences will be identical since $h_i(\overline{c}) = h_i(c)$. In case (b) each term in the second sequence will be the complex conjugate of the term in the first sequence, as can be shown by induction. (If $z'_n = \overline{z_n}$ then $z'_{n+1} = (\overline{z}_n)^k + h(\overline{c}) = \overline{(z_n)^k + h(c)} = \overline{z_{n+1}}$.) In either case, corresponding z terms have the same magnitude so c and $\overline{c}$ are either both in the fractal or both outside the fractal. Since $\overline{c}$ is the reflection of c in the real axis, the resulting fractal will be symmetric in the real axis.

For Result 2, set $c' = e^{ir}c$, corresponding to rotation by r radians. If the conditions of Result 2 hold, then induction shows that $z'_{2n} = e^{it}z_{2n}$ and $z'_{2n+1} = e^{is}z_{2n+1}$ for all n. (If $z'_{2n} = e^{it}z_{2n}$, then $z'_{2n+1} = (z'_{2n})^{k_1} + h_1(e^{ir}c) = e^{itk_1}(z_{2n})^{k_1} + e^{is}h_1(c) = e^{is}z_{2n+1}$. Likewise, if $z'_{2n+1} = e^{is}z_{2n+1}$, then $z'_{2n+2} = e^{it}z_{2n+2}$.) Thus $|z'_n| = |z_n|$ for all n, proving the resulting fractal is symmetric by rotation by r radians.

Result 3 is immediate, since $(e^{2\pi i/k_1}z_0)^{k_1} = e^{2\pi i}z_0^{k_1} = z_0^{k_1}$. Thus, the rotation of the initial z value has no effect after the first iteration.

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Figure captions:

Figure 1: $f_1(z)=z^3+c^2$, $f_2(z)=z^2+c^{-1}$

Figure 2: $f_1(z)=z^2+c^3$, $f_2(z)=z^3+\bar{c}$

Figure 3: $f_1(z)=z^6+c$, $f_2(z)=z^6+c/|c|$

Figure 4: Lyapunov exponents of the composite fractal $f_1(z)=z^3+\mathrm{Im}(c)$, $f_2(z)=z^2+\mathrm{Re}(c)$.

The bounds of c are $-1.31 \le x \le -1.025$, $.285 \le y \le 1.017$.

Pseudocode

```
function compute_lyapunov_pixel(x,y)
    z = .5
    lambda = 0
    for i from 1 to max do
        if i>skip then
            lambda = lambda + log(abs(3*z*z))   Note: d/dz z^3+y = 3z^2
        end if
        z = z*z*z+y;
        if i>skip then
            lambda = lambda + log(abs(2*z))     Note: d/dz z^2+x = 2z
        end if
        z = z*z+x;
    end for
    lambda = lambda / 2 / (max-skip)
    set_pixel(x,y,lambda)
```